



UNIVERSITY OF  
**LIVERPOOL**

## **JANUARY EXAMINATIONS 2010**

Bachelor of Science: Year 3  
Master of Physics: Year 3  
Master of Physics: Year 4

### **STATISTICAL AND LOW TEMPERATURE PHYSICS**

TIME ALLOWED: 3 hours

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#### INSTRUCTIONS TO CANDIDATES

Answer **all** questions.

Question 1 carries 50% of the total marks.

Questions 2 and 3 each carry 25% of the total marks.

Answer either part (a) or part (b) of questions 2 and 3.

In the event of a student answering both parts of an either/or question and not clearly crossing out one answer, only the answer to part (a) of the question will be marked.

The marks allotted to each part of a question are indicated in square brackets.

All symbols have their usual meanings unless otherwise stated.

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### Question 1.

- (a) There are  $n_1$  and  $n_2$  particles respectively in the energy levels 0 and  $\epsilon$ . A set of 2 distinguishable particles occupies these levels.
- State all possible macrostates of the particles in the form  $(n_1, n_2)$ . [3]
  - How many microstates are there for each of the macrostates. [3]
  - Write down the probability for each macrostate. Then find the mean populations of each energy level. [3]
  - If, instead of being distinguishable, the particles had been indistinguishable bosons, how many microstates are there for each of the macrostates? [3]

### Solution

(i) (0, 2), (1, 1), and (2, 0). [B3]

(ii) 1, 2 and 1 respectively. [B3]

(iii)

	levels		
	0	$\epsilon$	probability
macrostates	0	2	$\frac{1}{4}$
	1	1	$\frac{1}{2}$
	2	0	$\frac{1}{4}$

[B1]

Mean population for level 0 =  $0 \times \frac{1}{4} + 1 \times \frac{1}{2} + 2 \times \frac{1}{4} = 1$

Mean population for level 1 =  $2 \times \frac{1}{4} + 1 \times \frac{1}{2} + 0 \times \frac{1}{4} = 1$  [U2]

(iv) 1, 1 and 1 respectively. [U3]

- (b) In a solid state system the atoms can exist in energy states 0,  $\epsilon$ , or  $2\epsilon$ . Given that  $\epsilon = 1.38 \times 10^{-21}$  J, find the temperature in each of the following two cases:

- The population of the state of energy  $\epsilon$  is 0.1 times that of energy state 0. [3]
- The population of the state of energy  $2\epsilon$  is  $10^{-6}$  times that of energy state 0. [3]

If the temperature is then raised to 9945 K,

iii) how does the population of the state of energy  $2\varepsilon$  compare with that of energy state  $\varepsilon$ . [3]

Solution

(i) The ratio of the Boltzmann factor is

$$\exp(-\varepsilon/k_B T) = 0.1 \quad [B1]$$

Solving:

$$\begin{aligned} -\varepsilon / k_B T &= \ln 0.1 \\ T &= -\varepsilon / (k_B \ln 0.1) = 43.4 \text{ K} \end{aligned} \quad [B2]$$

(ii) The ratio of the Boltzmann factor is

$$\exp(-2\varepsilon/k_B T) = 10^{-6} \quad [B1]$$

Solving:

$$\begin{aligned} -2\varepsilon / k_B T &= \ln 10^{-6} \\ T &= -2\varepsilon / (k_B \ln 10^{-6}) = 14.5 \text{ K} \end{aligned} \quad [B2]$$

(iii) The energy difference between the two states is  $\varepsilon$ . So the ratio of populations is

given by the Boltzmann factor  $\exp(-\varepsilon/k_B T)$ . [B1]

Substituting the given  $\varepsilon$  and temperature:

$$\exp(-1.38 \times 10^{-21} / (k_B \times 9945)) = 0.99 \quad [B2]$$

(c) Consider spin-1/2 ions in a paramagnetic salt. The energy levels  $-\varepsilon$  and  $+\varepsilon$  are occupied by  $n_1$  and  $n_2$  particles respectively. The total number of particles is

$$N = n_1 + n_2$$

and the total energy is given by

$$U = -n_1 \varepsilon + n_2 \varepsilon.$$

i) By solving the two equations, show that

$$n_1 = \frac{N\varepsilon - U}{2\varepsilon} \quad \text{and} \quad n_2 = \frac{U + N\varepsilon}{2\varepsilon}. \quad [4]$$

ii) Given that  $U = -N\varepsilon/2$ , show that

$$n_1 = 3N/4 \quad \text{and} \quad n_2 = N/4. \quad [2]$$

iii) Using the Boltzmann factors  $\exp(+\epsilon/k_B T)$  and  $\exp(-\epsilon/k_B T)$ , show that

$$\exp(2\epsilon/k_B T) = 3$$

and obtain an expression for the temperature,  $T$ . [4]

### Solution

$$(i) \quad N = n_1 + n_2 \quad (1)$$

$$U = -n_1\epsilon + n_2\epsilon \quad (2)$$

$$\epsilon \times (1): \quad N\epsilon = n_1\epsilon + n_2\epsilon \quad (3)$$

$$(3) - (2): \quad N\epsilon - U = 2n_1\epsilon$$

$$\text{Rearranging:} \quad n_1 = \frac{N\epsilon - U}{2\epsilon} \quad [U2]$$

$$(3) + (2): \quad N\epsilon + U = 2n_2\epsilon$$

$$\text{Rearranging:} \quad n_2 = \frac{U + N\epsilon}{2\epsilon} \quad [U2]$$

(ii)

Substituting into

$$n_1 = \frac{N\epsilon - U}{2\epsilon}$$

from the previous question, we find

$$n_1 = \frac{N\epsilon - (-N\epsilon/2)}{2\epsilon}$$

Simplifying, we get

$$n_1 = \frac{3N}{4} \quad [U1]$$

Substituting into

$$n_2 = \frac{U + N\epsilon}{2\epsilon}$$

from the previous question, we find

$$n_2 = \frac{-N\epsilon/2 + N\epsilon}{2\epsilon}$$

Simplifying, we get

$$n_2 = \frac{N}{4} \quad [U1]$$

(iii)

The Boltzmann factor for level  $-\epsilon$  is  $\exp(+\epsilon / k_B T)$ .

The Boltzmann factor for level  $+\epsilon$  is  $\exp(-\epsilon / k_B T)$ .

The ratio is equal to the ratio of particle numbers:

$$\exp(+\epsilon / k_B T) : \exp(-\epsilon / k_B T) = n_1 : n_2 \quad [\text{U1}]$$

Using the results from the previous question:

$$\exp(+\epsilon / k_B T) : \exp(-\epsilon / k_B T) = (3N/4) : (N/4)$$

Simplifying, we get  $\exp\left(\frac{2\epsilon}{k_B T}\right) = 3.$  [U2]

Solving,  $T = 2\epsilon / (k_B \ln 3)$  [U1]

- (d) In a system of  $N$  conduction electrons contained in a volume  $V$ , the density of states  $g(k)$  in terms of the wavevector  $k$  is given by

$$g(k)dk = \frac{2 \cdot V \cdot 4\pi k^2 \cdot dk}{(2\pi)^3}.$$

- i) Describe what is meant by the Fermi energy of a system of electrons. [3]
- ii) Explain the origin of the 2 in the top line of the above expression. [1]
- iii) Show that at temperature  $T = 0$  K the  $k$  value at the Fermi surface, is given by

$$k_F = (3\pi^2 N / V)^{1/3}. \quad [3]$$

- iv) Write an expression relating  $k_F$  to the Fermi energy  $\epsilon_F$ . [1]
- v) Assuming that conduction electrons in metallic sodium (Na) can be represented by the theory above and that each Na atom releases one electron to the conduction band, evaluate the Fermi energy of sodium. [2]

Relative atomic mass of sodium = 23.0. Density of sodium =  $971 \text{ kg m}^{-3}$ .

**Solution**

(i) Because of the exclusion principle, no two electrons can occupy the same state.

[B1]

At 0 K, all the energy levels from the ground state until a certain level will be filled.

This highest energy level is the Fermi level.

[B2]

(ii) The factor 2 is because the electron can be spin up or spin down.

[B1]

(iii) At 0 K, the total number of particles is equal to the total number of states from  $k = 0$  to  $k_F$ :

$$N = \int_0^{k_F} g(k) dk \quad [B1]$$

Substituting the formula for  $g(k)$ ,

$$N = \int_0^{k_F} \frac{2.V.4\pi k^2 dk}{(2\pi)^3} \quad [B1]$$

Integrating,

$$N = \frac{2.V.4\pi k_F^3}{3(2\pi)^3}$$

Rearranging,

$$k_F = (3\pi^2 N / V)^{1/3} \quad [B1]$$

$$(iv) \quad \mathcal{E}_F = \frac{\hbar^2 k_F^2}{2m} \quad [B1]$$

(v)

The atomic mass is  $m = 23u$ .

The number density  $N/V$  is equal to the mass density divided by the atomic mass:

$$\begin{aligned} N/V &= 971 / (23u) \\ &= 2.542 \times 10^{28} \text{ m}^{-3} \end{aligned}$$

Then

$$k_F = (3\pi^2 N / V)^{1/3} = 9.097 \times 10^9 \text{ m}^{-1}$$

and the Fermi energy is

$$\mathcal{E}_F = \frac{\hbar^2 k_F^2}{2m} = 1.20 \times 10^{-23} \text{ J} \quad [B2]$$

(e) Phonons and electrons are two main contributors to the heat capacity in a metal.

i) At low temperature, the phonon heat capacity is given by

$$C_{ph} = 1944 \left( \frac{T}{\theta_D} \right)^3 \quad [\text{J mol}^{-1} \text{K}^{-1}],$$

where  $\theta_D$  is the Debye temperature. This may be written in the form  $AT^3$ .

Calculate  $A$  for silver. [The Debye temperature is 227 K.]

[2]

ii) At low temperature, the electron heat capacity is given by

$$C_e = \frac{\pi^2}{2} R \frac{T}{T_F} \quad [\text{J mol}^{-1} \text{K}^{-1}],$$

where  $T_F$  is the Fermi temperature. This may be written in the form  $\gamma T$ . Calculate

$\gamma$  for silver. [The Fermi temperature is  $6.36 \times 10^4$  K.]

[2]

iii) At what temperature is the phonon contribution equal to the electron contribution?

How do they compare below this temperature?

[3]

iv) The total heat capacity at low temperature is

$$C = \gamma T + AT^3.$$

This can be verified by measurement. Rewrite it in a form that is suitable for this purpose, and explain why it is suitable.

[2]

### Solution

(i)

Comparing  $C_{ph} = 1944 \left( \frac{T}{\theta_D} \right)^3$  with  $AT^3$ ,

gives

$$A = \frac{1944}{\theta_D^3} \quad [\text{B1}]$$

For silver,

$$A = \frac{1944}{227^3} = 1.662 \times 10^{-4} \text{ J mol}^{-1} \text{ K}^{-4} \quad [\text{B1}]$$

(ii)

Comparing  $C_e = \frac{\pi^2}{2} R \frac{T}{T_F}$  with  $\gamma T$ ,

gives

$$\gamma = \frac{\pi^2}{2} \frac{R}{T_F} \quad [B1]$$

For silver,

$$\gamma = \frac{\pi^2}{2} \frac{R}{6.36 \times 10^{-4}} = 6.451 \times 10^{-4} \text{ J mol}^{-1} \text{ K}^{-2} \quad [B1]$$

(iii)

$$AT^3 = \gamma T \quad [B1]$$

$$T^2 = \frac{\gamma}{A}$$

Using the previous results, we find  $T = 1.97 \text{ K}$  [B1]

Below this temperature, the phonon contribution falls as  $T^3$ . So it would be much smaller  
compare with the electronic contribution which falls as  $T$ . [B1]

(iv)  $\frac{C}{T} = \gamma + AT^2$  [B1]

Plotting  $C/T$  against  $T^2$  gives a straight line. [B1]

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**Question 2.** Answer either (a) or (b)

- 2(a) A Maxwell-Boltzmann gas consists of  $N$  monatomic particles in a box with rigid walls. The box is in the shape of a cube with side length  $L$ , with walls at  $x = L$ ,  $y = L$  and  $z = L$ . The wave function for a single particle can be written as:

$$\psi = A \sin(k_x x) \sin(k_y y) \sin(k_z z),$$

where  $A$  is a constant, and  $k_x$ ,  $k_y$  and  $k_z$  are positive.

- i) Show that possible states have wave numbers  $k_x$  that satisfies:

$$k_x = \frac{\pi}{L} n_x.$$

where  $n_x$  is an integer. Why must  $n_x$  be positive?

[3]

Similarly the wave numbers  $k_y$  and  $k_z$  satisfy  $k_y = \frac{\pi}{L} n_y$  and  $k_z = \frac{\pi}{L} n_z$ .

- ii) The wave number  $k = \sqrt{k_x^2 + k_y^2 + k_z^2}$ . Using the above results, show that

$$\left( \frac{kL}{\pi} \right)^2 = n_x^2 + n_y^2 + n_z^2.$$

[2]

**Solution**

(i)

At  $x = L$ ,  $\psi = 0$

So  $\sin(k_x L) = 0$  [B1]

Therefore  $k_x L = n_x \pi$  or  $k_x = \frac{\pi}{L} n_x$

where  $n_x$  is an integer. [B1]

$n_x$  must be positive because  $k_x$  is positive. [B1]

(ii)

Square the wave number:  $k^2 = k_x^2 + k_y^2 + k_z^2$

Substitute the previous results:  $k^2 = \left( \frac{\pi}{L} n_x \right)^2 + \left( \frac{\pi}{L} n_y \right)^2 + \left( \frac{\pi}{L} n_z \right)^2$  [B1]

Rearrange:

$$\left(\frac{kL}{\pi}\right)^2 = n_x^2 + n_y^2 + n_z^2 \quad [\text{B1}]$$

iii) Explain why states with wave number less than  $k$  are enclosed in a sphere of radius

$$R = \frac{kL}{\pi}. \quad [2]$$

iv) Find the number of points in the sphere. Show that the number of states with wave number less than  $k$  is

$$G(k) = \frac{1}{8} \times \frac{4\pi}{3} \left(\frac{kL}{\pi}\right)^3. \quad [3]$$

v) Show that the number of states with wave number between  $k$  and  $k+dk$  is given by:

$$g(k)dk = \frac{V}{2\pi^2} k^2 dk,$$

where  $V = L^3$  is the volume of the box. [2]

solution

(iii)

For states with wave number less than  $k$ , the points would be inside a sphere of radius  $R$ . [B1]

Compare the previous result

$$\left(\frac{kL}{\pi}\right)^2 = n_x^2 + n_y^2 + n_z^2$$

with the equation for a sphere

$$R^2 = n_x^2 + n_y^2 + n_z^2$$

The radius is

$$R = \frac{kL}{\pi}. \quad [\text{B1}]$$

(iv)

There is one point per unit volume. So the total number of states is the volume of the sphere  $\frac{4}{3}\pi R^3$ . [B1]

Because  $(n_x, n_y, n_z)$  are positive, the points would be in 1/8 of the sphere only. So the total number of states is

$$G(k) = \frac{1}{8} \times \frac{4\pi}{3} R^3 \quad [\text{B1}]$$

Substituting the above result for  $R$  gives

$$G(k) = \frac{1}{8} \times \frac{4\pi}{3} \left( \frac{kL}{\pi} \right)^3 \quad [B1]$$

(v)

When  $k$  is increased by  $dk$ , the number of states below  $k$  changes by

$$dG(k) = \frac{dG(k)}{dk} dk$$

This is just the  $g(k)dk$  requested by the question. [B1]

Differentiating the above formula for  $G(k)$ , we get

$$g(k)dk = \frac{L^3}{2\pi^2} k^2 dk$$

Replacing  $L^3$  with  $V$ , we get the answer. [B1]

vi) Show that the number of states with energy between  $\epsilon$  and  $\epsilon + d\epsilon$  is given by:

$$g(\epsilon)d\epsilon = \frac{V}{4\pi^2} \left( \frac{2m}{\hbar^2} \right)^{\frac{3}{2}} \sqrt{\epsilon} d\epsilon. \quad [4]$$

Solution

The intervals  $d\epsilon$  and  $dk$  refer to the same number of states, so

$$g(\epsilon)d\epsilon = g(k)dk$$

or

$$g(\epsilon)d\epsilon = \frac{L^3}{2\pi^2} k^2 dk \quad [B1]$$

To express in terms of  $\epsilon$ ,

make  $k$  the subject:  $k = \sqrt{\frac{2m\epsilon}{\hbar^2}}$  [B1]

Find  $dk$ :

$$dk = \sqrt{\frac{2m}{\hbar^2}} \frac{1}{2} \epsilon^{-1/2} d\epsilon \quad [B1]$$

Substitute both:

$$g(\epsilon)d\epsilon = \frac{L^3}{2\pi^2} \times \frac{2m\epsilon}{\hbar^2} \times \sqrt{\frac{2m}{\hbar^2}} \frac{1}{2} \epsilon^{-1/2} d\epsilon$$

Simplify

$$g(\varepsilon)d\varepsilon = \frac{V}{4\pi^2} \left( \frac{2m}{\hbar^2} \right)^{\frac{3}{2}} \sqrt{\varepsilon} d\varepsilon$$

[B1]

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2(a) (continued)

vii) The partition function is given by

$$Z = \frac{V}{8} \left( \frac{2mk_B T}{\pi \hbar^2} \right)^{3/2}.$$

Explain why the number of particles with energy between  $\epsilon$  and  $\epsilon+d\epsilon$  is given by the Boltzmann distribution:

$$n(\epsilon) d\epsilon = \frac{N}{Z} g(\epsilon) \exp\left(-\frac{\epsilon}{k_B T}\right) d\epsilon. \quad [3]$$

Solution

The probability that a state is occupied is  $\frac{\exp(-\epsilon/k_B T)}{Z}$ . [B1]

The mean population of the state is  $N \frac{\exp(-\epsilon/k_B T)}{Z}$  [B1]

The number of states in the energy interval is  $g(\epsilon) d\epsilon$

So the number of particles is the product of all of the above. [B1]

viii) Using the previous results, show that the number of particles in the Maxwell-Boltzmann gas with energy between  $\epsilon$  and  $\epsilon+d\epsilon$  can be written:

$$n(\epsilon) d\epsilon = N \frac{2}{\sqrt{\pi}} (k_B T)^{-3/2} \sqrt{\epsilon} \exp\left(-\frac{\epsilon}{k_B T}\right) d\epsilon. \quad [2]$$

Solution

Substituting the formulae for  $g(\epsilon)$  and  $Z$  into

$$n(\epsilon) d\epsilon = \frac{N}{Z} g(\epsilon) \exp\left(-\frac{\epsilon}{k_B T}\right) d\epsilon \quad [B1]$$

we get

$$n(\epsilon) d\epsilon = N \div \left[ \frac{V}{8} \left( \frac{2mk_B T}{\pi \hbar^2} \right)^{3/2} \right] \times \frac{V}{4\pi^2} \left( \frac{2m}{\hbar^2} \right)^{3/2} \sqrt{\epsilon} \times \exp\left(-\frac{\epsilon}{k_B T}\right) d\epsilon$$

Simplifying gives the answer. [B1]

ix) Show that the total energy is given by

$$U = \frac{3}{2} N k_B T . \quad [4]$$

You are given that:

$$\int_0^{\infty} x^{3/2} \exp\left(-\frac{x}{a}\right) dx = \frac{3\sqrt{\pi}}{4} a^{5/2} .$$

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Solution

Substitute the previous formula for  $n(\epsilon)$ :

$$U = \int_0^{\infty} N \frac{2}{\sqrt{\pi}} (k_B T)^{-3/2} \sqrt{\epsilon} \exp\left(-\frac{\epsilon}{k_B T}\right) \times \epsilon d\epsilon \quad [B1]$$

Let  $x = \epsilon$ ,  $a = k_B T$ . Substituting:

$$U = N \frac{2}{\sqrt{\pi}} (k_B T)^{-3/2} \int_0^{\infty} x^{3/2} \exp\left(-\frac{x}{a}\right) dx \quad [B1]$$

Using the given integral formula:

$$U = N \frac{2}{\sqrt{\pi}} (k_B T)^{-3/2} \times \frac{3\sqrt{\pi}}{4} a^{5/2} \quad [B1]$$

Substitute  $a = k_B T$ :

$$U = N \frac{2}{\sqrt{\pi}} (k_B T)^{-3/2} \times \frac{3\sqrt{\pi}}{4} (k_B T)^{5/2}$$

Simplifying, we get

$$U = \frac{3}{2} N k_B T \quad [B1]$$

2 (b)

A solid cube of length  $a$  has one corner at the origin and three of its edges along the axes  $x$ ,  $y$  and  $z$ . The vibration in the solid is described by the wave equation:

$$\frac{\partial^2 \xi}{\partial x^2} + \frac{\partial^2 \xi}{\partial y^2} + \frac{\partial^2 \xi}{\partial z^2} = \frac{1}{v^2} \frac{\partial^2 \xi}{\partial t^2},$$

where  $\xi$  is related to the displacement of the atoms, and  $v$  is the velocity of sound.

i) Explain why we only need to consider positive  $k_x$ .

[2]

Solution

$$\xi = e^{i\omega t} \sin k_x x \sin k_y y \sin k_z z$$

Replace  $k_x$  by  $-k_x$ :

$$\xi = e^{i\omega t} \sin(-k_x x) \sin k_y y \sin k_z z$$

This is negative of the original  $\xi$ .

[U1]

It is just multiplied by a constant factor. So it is the same solution.

Therefore we only need to consider  $+k_x$ .

[U1]

ii) Show using the trial solution that  $k_x$  must be quantised as follows:

$$k_x = \frac{n_x \pi}{a},$$

where  $n_x$  is a positive integer.

[2]

Solution

At  $x = a$ ,  $\xi = 0$

So  $\sin(k_x a) = 0$

[B1]

Therefore  $k_x a = n_x \pi$  or  $k_x = \frac{\pi}{a} n_x$

where  $n_x$  is a positive integer.

[B1]

Similarly, it can be shown that

$$k_y = \frac{n_y \pi}{a} \quad \text{and} \quad k_z = \frac{n_z \pi}{a},$$

where  $n_y$  and  $n_z$  are positive integers.

iii) The total number of states with frequency less than  $\omega$  is given by

$$G(\omega) = 3 \times \frac{\pi}{6} \left( \frac{\omega a}{v\pi} \right)^3.$$

Explain why the extra factor of 3 is needed.

[2]

Solution

For each frequency,

there are 2 transverse polarisations and

[B1]

1 longitudinal.

[B1]

iv) Show that the density of states is

$$g(\omega) = 3 \times \frac{a^3 \omega^2}{2\pi^2 v^3}.$$

[2]

Solution

$$g(\omega) = \frac{dG(\omega)}{d\omega}$$

[B1]

Differentiating the formula for  $G(\omega)$ :

$$g(\omega) = 3 \times \frac{\pi}{6} \left( \frac{a}{v\pi} \right)^3 3\omega^2$$

[B1]

Next, consider a 2-D cube. This would be a square of side  $a$ . Since there are only two dimensions, we may leave out  $k_z$  from the above equations.  $k_x$  and  $k_y$  are related by

$$k_x^2 + k_y^2 = \frac{\omega^2}{v^2}.$$

v) The total number of states with frequency less than  $\omega$  may be estimated by counting the number of points  $(n_x, n_y)$  enclosed within a quadrant of a circle in a 2-D. Show that the radius of this circle is

$$R = \frac{a\omega}{v\pi}.$$

[2]



Solution

Given that  $k_x^2 + k_y^2 = \frac{\omega^2}{v^2}$

Substituting results for k,  $\left(\frac{n_x \pi}{a}\right)^2 + \left(\frac{n_y \pi}{a}\right)^2 = \frac{\omega^2}{v^2}$

Rearranging  $n_x^2 + n_y^2 = \left(\frac{a\omega}{v\pi}\right)^2$  [B1]

The equation for a circle is  $n_x^2 + n_y^2 = R^2$

Comparing, the radius is  $R = \frac{a\omega}{v\pi}$  [B1]

vi) Show that the number of states is

$$G(\omega) = 2 \times \frac{\pi}{4} \left( \frac{\omega a}{v\pi} \right)^2. \quad [3]$$

Solution

There is one point per unit area.

So the number of states is the area of the quadrant times 2:

$$G(\omega) = 2 \times \frac{\pi R^2}{4} = 2 \times \frac{\pi}{4} \left( \frac{\omega a}{v\pi} \right)^2 \quad [B1]$$

The factor of 2 is because for each frequency in 2-D, there would be

1 transverse polarisation and [B1]

1 longitudinal. [B1]

2(b) (continued)

vii) Show that the density of states is

$$g(\omega) = 2 \times \frac{a^2 \omega}{2\pi v^2}. \quad [1]$$

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### Solution

Differentiating the formula for  $G(\omega)$ :

$$g(\omega) = 2 \times \frac{\pi}{4} \left( \frac{a}{v\pi} \right)^2 2\omega^2 = 2 \times \frac{a^2 \omega}{2\pi v^2} \quad [\text{B1}]$$


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Consider a 2-D solid. The phonons obey the Bose-Einstein distribution

$$f(\omega) = \frac{1}{\exp(\hbar\omega/k_B T) - 1}$$

where  $T$  is the temperature of the solid, the energy of each phonon is  $\hbar\omega$ .

viii) Explain how the following expression for energy is obtained:

$$U = \int_0^{\omega_p} \frac{\hbar\omega g(\omega) d\omega}{\exp(\hbar\omega/k_B T) - 1}.$$

where  $T$  is the temperature of the solid, the energy of each phonon is  $\hbar\omega$ . [3]

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### Solution

The total energy is  $U = \int \hbar\omega n(\omega) d\omega$  [B1]

The number density is obtained by multiplying density of states with the occupation function:  $n(\omega) d\omega = g(\omega) f(\omega) d\omega$  [B1]

### Substituting

$$U = \int \hbar\omega g(\omega) f(\omega) d\omega = \int \frac{\hbar\omega g(\omega) d\omega}{\exp(\hbar\omega/k_B T) - 1} \quad [\text{B1}]$$


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ix) Why is there an upper limit  $\omega_p$ ? [1]

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### Solution

There is an upper limit because the number of possible vibration modes depends on the number of particles. [B1]

The energy is given by

$$U = \frac{a^2 \hbar}{\pi v^2} \int_0^{\omega_p} \frac{\omega^2 d\omega}{\exp(\hbar\omega/k_B T) - 1}.$$

- x) Use the substitution  $x = \hbar\omega/k_B T$  to show that at low temperature, the energy approaches the following form:

$$U = \frac{a^2 k_B^3 T^3}{\pi v^2 \hbar^2} \int_0^\infty \frac{x^2 dx}{e^x - 1}. \quad [4]$$

Solution

Let  $x = \hbar\omega/k_B T$  [B1]

and  $x_D = \hbar\omega_D/k_B T$  [B1]

Small T means large  $x_D$ .

Substitute  $\omega = k_B T x / \hbar$ :

$$U = \frac{a^2 \hbar}{\pi v^2} \int_0^{\omega_p} \frac{\omega^2 d\omega}{\exp(\hbar\omega/k_B T) - 1} = \frac{a^2 \hbar}{\pi v^2} \left( \frac{k_B T}{\hbar} \right)^3 \int_0^{x_D} \frac{x^2 dx}{e^x - 1} \quad [B1]$$

Approximate  $x_D$  to infinity:

$$U = \frac{a^2 \hbar}{\pi v^2} \left( \frac{k_B T}{\hbar} \right)^3 \int_0^\infty \frac{x^2 dx}{e^x - 1} \quad [B1]$$

- xi) Hence, show that the heat capacity  $C_V$  is proportional to  $T^2$ . [1]

Solution

The previous formula for U is proportional to  $T^3$ .

Differentiating gives  $T^2$ .

Since  $C = \frac{dU}{dT},$

C is proportional to  $T^2$ .

[B1]

- xii) In a 3-D solid,  $C_V$  is known to be proportional to  $T^3$ . In graphite, it is found that  $C_V$  is proportional to  $T^{2.4}$ . Explain this behaviour. [2]

Solution

Graphite has a layered structure, with strong bonds between atoms within each layer and weak bonds between layers.

[B1]

As a result, phonons are mainly localised within each 2D layer.

[B1]

**Question 3.** Answer either (a) or (b)

3(a)

The two phases of liquid helium-3 in a dilution refrigerator may be described by the Fermi gas. The Fermi temperature  $T_F$  of a Fermi gas is defined as  $E_F/k_B$ . The Fermi energy  $E_F$  is given by

$$E_F = \frac{\hbar^2}{2m} \left( \frac{3\pi^2 N}{V} \right)^{2/3},$$

where  $m$  is the mass of each particle,  $N$  is the number of particles and  $V$  is the volume of the gas. The effective mass ratios in the dilute and concentrated phases are 2.45 and 2.78 respectively. Calculate the effective mass of the helium-3 atom in

i) the concentrated phase, and

[2]

ii) the dilute phase.

[1]

**Solution**

(i)

Mass of helium atom is  $m = 3u$ .

[U1]

$$m^* = 2.78 \times m = 2.78 \times 3 \times 1.6605 \times 10^{-27} \text{ kg} = 1.385 \times 10^{-26} \text{ kg}$$

[U1]

(ii)

$$m^* = 2.45m = 2.45 \times 1.6605 \times 3 \times 10^{-27} \text{ kg} = 1.220 \times 10^{-26} \text{ kg}$$

[U1]

The molar volume  $V_m$  of liquid helium-3 and liquid helium-4 are  $36.84 \text{ cm}^3$  and  $27.58 \text{ cm}^3$  respectively. Calculate the number density  $N/V$  of helium-3 atoms in

iii) the concentrated phase, and

[2]

iv) the dilute phase. Assume that the dilute phase has a concentration of 6.6%.

[3]

**Solution**

(iii)

Consider one mole of pure helium-3 [U1]

The volume  $V_m$  is  $36.84 \text{ cm}^3$ .

There are  $N_A$  atoms.

$$N/V = N_A / V_m$$

$$= 6.022 \times 10^{23} / 36.84 \times 10^{-6} \text{ m}^3 = 1.635 \times 10^{28} \text{ m}^{-3} \quad [\text{U1}]$$

(iv)

Consider one mole of the dilute phase.

The volume  $V_m$  is approximately that of pure helium-4,  $27.58 \text{ cm}^3$ . [U1]

There are  $N_A$  atoms.

6.6% of these are helium-3. [U1]

$$N/V = 0.066 N_A / V_m$$

$$= 0.066 \times 6.022 \times 10^{23} / 27.58 \times 10^{-6} \text{ m}^3 = 1.441 \times 10^{27} \text{ m}^{-3} \quad [\text{U1}]$$

Using the above results, calculate the Fermi energy and Fermi temperature of helium-3 atoms in

v) the concentrated phase, and [2]

vi) the dilute phase. [2]

Solution

(v)

Previous results give

$$m^* = 1.385 \times 10^{-26} \text{ kg}$$

$$N/V = 1.635 \times 10^{28} \text{ m}^{-3}$$

Substituting,

$$E_F = \frac{\hbar^2}{2m^*} \left( \frac{3\pi^2 N}{V} \right)^{2/3} = 1.154 \times 10^{-23} \text{ J} \quad [\text{U2}]$$

(vi)

Previous results give

$$m^* = 1.220 \times 10^{-26} \text{ kg}$$

$$N/V = 1.441 \times 10^{27} \text{ m}^{-3}$$

Substituting,

$$E_F = \frac{\hbar^2}{2m^*} \left( \frac{3\pi^2 N}{V} \right)^{2/3} = 2.594 \times 10^{-24} \text{ J} \quad [\text{U2}]$$


---

The heat capacity per mole of liquid helium-3 is

$$C = \frac{\pi^2}{2} \frac{T}{T_F} R.$$

vii) Using the following relations for the entropy  $S$  and the heat  $Q$ :

$$S = \int \frac{dQ}{T} \quad \text{and} \quad C = \frac{dQ}{dT},$$

show that

$$S = \int_0^T \frac{C dT'}{T'}. \quad [2]$$

viii) By integrating this, show that the entropy per mole of liquid helium-3 is equal to its heat capacity. [2]

---

**Solution**

(vii)

Rearrange  $dQ = C dT$  [U1]

Substitute into  $S = \int \frac{dQ}{T}$

Gives  $S = \int_0^T \frac{C dT'}{T'}$  [U1]

(viii)

Substitute the formula for  $C$  into  $S = \int \frac{dQ}{T}$

we get

$$S = \int_0^T \frac{\pi^2}{2} \frac{T'}{T_F} R dT' \quad \div \quad T = \int_0^T \frac{\pi^2}{2} \frac{1}{T_F} R dT' \quad [\text{U1}]$$

Integrating gives

$$S = \frac{\pi^2}{2} \frac{T}{T_F} R$$

which is the same as the heat capacity  $C$ . [U1]

---

The helium-3 in the two phases in the dilution refrigerator are approximately in equilibrium. So the chemical potentials of the phases are equal:

$$H_C - TS_C = H_D - TS_D,$$

where  $H$  is the enthalpy per mole. The subscripts  $D$  and  $C$  denote the dilute phase (6.6% concentration) and the concentrated phase respectively.

ix) Show that the enthalpy change in the dilution refrigerator is given by

$$\Delta H = T(S_D - S_C). \quad [2]$$

x) Explain why this can be written as

$$\Delta H = T(C_D - C_C),$$

where  $C_D$  and  $C_C$  are the heat capacities. [2]

---

**Solution**

(ix)

Rearranging

$$H_D - H_C = TS_D - TS_C \quad [\text{U1}]$$

The left hand side is the enthalpy change,

so we get

$$\Delta H = T(S_D - S_C) \quad [\text{U1}]$$

(x)

The entropy is equal to the heat capacity. [U1]

So  $C_D = S_D$  and  $C_C = S_C$

Therefore  $\Delta H = T(S_D - S_C) = T(C_D - C_C)$  [U1]

---

xi) Using the above formula for heat capacity, show that



$$\Delta H = \frac{\pi^2}{2} \left( \frac{1}{T_D} - \frac{1}{T_C} \right) T^2 R,$$

where  $T_D$  and  $T_C$  are the Fermi temperatures of the dilute phase and the concentrated phase.

[3]

- xii) Given that the Fermi temperatures are 0.4 K and 1.8 K respectively, show that the enthalpy change is

$$\Delta H = 80T^2 \quad [\text{J/mol}].$$

[2]

**Solution**

(xi)

The heat capacities are

$$C_C = \frac{\pi^2}{2} \frac{T}{T_C} R$$

$$C_D = \frac{\pi^2}{2} \frac{T}{T_D} R$$

[U1]

Substituting into  $\Delta H = T(C_D - C_C)$

we get 
$$\Delta H = T \left( \frac{\pi^2}{2} \frac{T}{T_D} R - \frac{\pi^2}{2} \frac{T}{T_C} R \right)$$

Rearranging gives the answer.

[U2]

(xii)

$$T_D = 0.4 \text{ K}$$

$$T_C = 1.8 \text{ K}$$

Substitute into 
$$\Delta H = \frac{\pi^2}{2} \left( \frac{1}{T_D} - \frac{1}{T_C} \right) T^2 R$$

Get 
$$\Delta H = 79.8T^2 \quad [\text{J/mol}]$$

[U2]

3 (b)

Cerium Magnesium Nitrate (CMN) is a spin-1/2 paramagnetic salt that is used in magnetic cooling.

- i) The magnetic heat capacity of the spin-1/2 salt at a temperature  $T$  and a magnetic field  $B$  is given by

$$C = Nk_B x^2 \operatorname{sech}^2 x,$$

where  $x = \frac{\mu_B B}{k_B T}$ . Show that the high temperature approximation is given by

$$C_a = Nk_B x^2.$$

For this to be valid, how should  $k_B T$  compare with  $\mu_B B$ ?

[3]

**Solution**

High  $T$  means small  $x$ .

[U1]

$x \rightarrow 0$ ,  $\operatorname{sech} x \rightarrow 1$

So  $C \rightarrow Nk_B x^2$

[U1]

$x = \frac{\mu_B B}{k_B T}$ . Small  $x$  means that  $k_B T$  must be much larger than  $\mu_B B$ .

[U1]

- ii) Show that the approximate heat capacity is always greater than the exact one.

[2]

**Solution**

Exact  $C = Nk_B x^2 \operatorname{sech}^2 x$

Approximate  $C = Nk_B x^2$

For positive  $x$ ,  $\operatorname{sech} x < 1$ .

So  $C_a > C$ .

[U2]

- iii) For a magnetic field of 1 T, at what temperature does the high temperature approximation for the heat capacity deviate by 5% from the exact results. [3]

-----

Solution

The difference is  $Nk_B x^2 - Nk_B x^2 \operatorname{sech}^2 x$

When this is 5% of the exact result, we have

$$Nk_B x^2 - Nk_B x^2 \operatorname{sech}^2 x = 0.05 Nk_B x^2 \operatorname{sech}^2 x \quad [\text{U1}]$$

Simplifying  $\cosh x = \sqrt{1.05}$   
 $x = 0.2218$

To find T,

rearrange  $x = \frac{\mu_B B}{k_B T}$

Given B = 1 T.

So  $T = \frac{\mu_B B}{k_B x} = 3.09 \text{ K} \quad [\text{U2}]$

- 
- iv) 75% of the paramagnetic ions are polarised with their spin in one direction, and the rest in the opposite direction. Show that

$$e^{-2x} = 0.75. \quad [\text{2}]$$

-----

Solution

The Boltzmann factors are  $e^x$  and  $e^{-x}$ .

The ratio is 75%.

So  $e^{-x} / e^x = 0.75$

$$e^{-2x} = 0.75. \quad [\text{U2}]$$

- 
- v) To what temperature does one have to refrigerate the salt so that this would happen. [3]

-----

Solution

Solve  $e^{-2x} = 0.75$

Get  $x = -0.5 \ln 0.75$  [U1]

To find T,

rearrange  $x = \frac{\mu_B B}{k_B T}$

Given B = 1 T.

So  $T = \frac{\mu_B B}{k_B x} = 4.67 \text{ K}$  [U2]

A sample of CMN is initially at a magnetic field of 2 T and a temperature of 1 K.

vi) It is demagnetised to 0.003 T. During demagnetisation, the temperature is proportional to field. Calculate the final temperature. [2]

**Solution**

Temperature is proportional to field means

$$T_f / T_i = B_f / B_i$$

$$B_i = 2 \text{ T}$$

$$T_i = 1 \text{ K}$$

$$B_f = b \quad [U1]$$

$$\text{So } T_f = (B_f / B_i) T_i = (b/2) \times 1 = 0.0015 \text{ K} \quad [U1]$$

The magnetic energy in a spin  $1/2$  salt is given by

$$U = -N\mu_B B \tanh x,$$

where  $N$  is the number of particles.

vii) Find the magnetic energy in one mole of the salt, at a final field of 3 mT and final temperature of 1 mK. [2]

**Solution**

$$B = 0.003 \text{ T}$$

$$T = 0.001 \text{ K}$$

$$N = N_A$$

$$U = -N_A \mu_B B \tanh \frac{\mu_B B}{k_B T} = -0.01617 \text{ J} \quad [\text{U2}]$$

viii) Find the limiting energy at high temperature. [2]

Solution

$$T \rightarrow \infty, \quad x \rightarrow 0$$

$$\text{So } U \rightarrow 0 \quad [\text{U2}]$$

ix) The cooling power of the CMN is the heat that it would absorb when it warms up, which is converted to the energy. Calculate the cooling power of one mole of CMN. [2]

Solution

The heat is converted to the energy U.

$$\begin{aligned} \text{Heat absorbed} &= \text{energy at high } T - \text{energy at } 1 \text{ mK} \\ &= 0 - (-0.01617) \\ &= 0.01617 \text{ J} \end{aligned} \quad [\text{U2}]$$

The magnetic entropy in a spin-1/2 salt is given by

$$S = Nk_B \ln(2 \cosh x) - Nk_B x \tanh x .$$

x) State the entropy at a temperature of 1 K and zero field for one mole of the salt. Calculate the entropy when the field is increased to 2 T. [3]

Solution

For one mole,  $N = N_A$

1 K and zero field:  $x = 0$ ,  $S = Nk_B \ln 2$  [U1]

1 K and 2 T:  $x = \frac{\mu_B B}{k_B T} = 1.343$

$$S = N_A k_B \ln(2 \cosh x) - N_A k_B x \tanh x = 1.97 \text{ J / K} \quad [\text{U2}]$$

xi) For a reversible change, the heat absorbed is related to the entropy change by

$$dQ = T dS.$$

Calculate the heat of magnetisation which has to be removed if this salt is magnetised isothermally from 0 to 2 T.

[1]

Solution

From previous question,

$$dS = 1.97 \text{ J / K}$$

$$T = 1 \text{ K}$$

So heat of magnetisation is  $dQ = T dS = 1 \times 1.97 = 1.97 \text{ J}$  [U1]

# CONSTANTS

Speed of light in vacuum	$c$	$=$	$3.00 \times 10^8 \text{ ms}^{-1}$
Permeability of vacuum	$\mu_0$	$=$	$4\pi \times 10^{-7} \text{ Hm}^{-1}$
		$=$	$4\pi \times 10^{-7} \text{ VsA}^{-1}\text{m}^{-1}$
Permittivity of vacuum	$\epsilon_0$	$=$	$8.85 \times 10^{-12} \text{ Fm}^{-1}$
		$=$	$8.85 \times 10^{-12} \text{ AsV}^{-1}\text{m}^{-1}$
Elementary charge	$e$	$=$	$1.60 \times 10^{-19} \text{ C}$
Planck constant	$h$	$=$	$6.63 \times 10^{-34} \text{ Js}$
	$h/2\pi = \hbar$	$=$	$1.05 \times 10^{-34} \text{ Js}$
Avogadro constant	$N_A$	$=$	$6.02 \times 10^{23} \text{ mol}^{-1}$
Boltzmann constant	$k$	$=$	$1.38 \times 10^{-23} \text{ JK}^{-1}$
Gas constant	$R$	$=$	$8.31 \text{ JK}^{-1}\text{mol}^{-1}$
Unified atomic mass constant	$m_u$	$=$	$1.66 \times 10^{-27} \text{ kg}$
		$=$	$931.5 \text{ MeVc}^{-2}$
Electron mass	$m_e$	$=$	$9.11 \times 10^{-31} \text{ kg}$
Proton mass	$m_p$	$=$	$1.67 \times 10^{-27} \text{ kg}$
Gravitational constant	$G$	$=$	$6.67 \times 10^{-11} \text{ Nm}^2\text{kg}^{-2}$
Acceleration due to gravity	$g$	$=$	$9.8 \text{ ms}^{-2}$
Bohr magneton	$\mu_B$	$=$	$9.27 \times 10^{-24} \text{ JT}^{-1}$